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Using Differentiation under Fractional Integral Sign to Solve Three Types of Fractional Integrals

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional integral and a new multiplication of fractional analytic functions, we can find three types of fractional integrals. In fact, our results are generalizations of classical calculus results.**

Keywords: **Jumarie's modified R-L fractional integral, new multiplication, fractional analytic functions, fractional integrals.**

I. INTRODUCTION

Fractional calculus belongs to the field of mathematical analysis, involving the research and applications of arbitrary order integrals and derivatives. Fractional calculus originated from a problem put forward by L'Hospital and Leibniz in 1695. Therefore, the history of fractional calculus was formed more than 300 years ago, and fractional calculus and classical calculus have almost the same long history. Since then, fractional calculus has attracted the attention of many contemporary great mathematicians, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the efforts of researchers, the theory of fractional calculus and its applications have developed rapidly. On the other hand, fractional calculus has wide applications in physics, electrical engineering, , viscoelasticity, control theory, economics, and other fields [1-10].

However, the definition of fractional derivative is not unique. Commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, conformable fractional derivative, Jumarie's modified R-L fractional derivative [11-15]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with ordinary calculus.

In this paper, based on Jumarie type of R-L fractional integral and a new multiplication of fractional analytic functions, we use differentiation under fractional integral sign to evaluate the following three types of fractional integrals:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(Ln_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}p}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}t}\otimes_{\alpha}E_{\alpha}(x^{\alpha})\right],$$

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(Ln_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}p}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}t}\otimes_{\alpha}cos_{\alpha}(x^{\alpha})\right],$$

and

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(Ln_{\alpha}(x^{\alpha})\right)^{\otimes_{\alpha}p}\otimes_{\alpha}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}t}\otimes_{\alpha}sin_{\alpha}(x^{\alpha})\right],$$

where $0 < \alpha \leq 1$, p is a positive integer, and t is any real number. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([16])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \tag{1}$$

where $\Gamma(\ \ )$ is the gamma function. And the Jumarie type of R-L α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ . \tag{2}$$

Proposition 2.2 ([17])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{4}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([18])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced.

Definition 2.4 ([19])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \tag{6}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{8}$$

Definition 2.5 ([20])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([21])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{13}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha(g_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{a_n}{n!}(g_\alpha(x^\alpha))^{\otimes_\alpha n}, \tag{15}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha(f_\alpha(x^\alpha))=\sum_{n=0}^{\infty}\frac{b_n}{n!}(f_\alpha(x^\alpha))^{\otimes_\alpha n}. \tag{16}$$

Definition 2.7 ([22])**:** Let $0<\alpha\leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha\circ g_\alpha)(x^\alpha)=(g_\alpha\circ f_\alpha)(x^\alpha)=\frac{1}{\Gamma(\alpha+1)}x^\alpha. \tag{17}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.8 ([23])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{18}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and α-fractional sine function are defined as follows:

$$cos_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}, \tag{19}$$

and

$$sin_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{20}$$

Notation 2.9: If m,k,p are nonnegative integers, $k\leq m$, r is a real number. Let $\binom{m}{k}=\frac{m!}{k!(m-k)!}$, and let $(r)_p=r(r-1)\cdots(r-p+1)$ for all $p\geq 1$, and $(r)_0=1$.

Theorem 2.10 ([24])(differentiation under fractional integral sign)**:** *If* $0<\alpha\leq 1$, t *is a nonzero real variable, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function at* $x=0$, *then*

$$\frac{d}{dt}\left({}_0I_x^\alpha\right)[f_\alpha(t,x^\alpha)]=\left({}_0I_x^\alpha\right)\left[\frac{d}{dt}f_\alpha(t,x^\alpha)\right]. \tag{21}$$

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III. MAIN RESULTS

In this section, we find three types of fractional integrals by using differentiation under fractional integral sign.

Theorem 3.1: *Suppose that* $0 < \alpha \le 1$, p *is any integer,* t *is a real number, and* t *is not a nonnegative integer. Then*

$$\left({}_0I_x^\alpha\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{p}\binom{p}{k}(-1)_{p-k}\frac{1}{(n+t+1)^{p-k+1}}\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t+1)}. \tag{22}$$

Proof Since $\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]$

$$=\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!(n+t+1)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t+1)}. \tag{23}$$

It follows from differentiation under fractional integral sign that

$$\frac{d^p}{dt^p}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\frac{d^p}{dt^p}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\frac{d^p}{dt^p}\left[E_\alpha\left(tLn_\alpha(x^\alpha)\right)\right]\otimes_\alpha E_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]. \tag{24}$$

Therefore,

$$\left({}_0I_x^\alpha\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]$$

$$=\frac{d^p}{dt^p}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha E_\alpha(x^\alpha)\right]$$

$$=\frac{d^p}{dt^p}\left[\sum_{n=0}^{\infty}\frac{1}{n!(n+t+1)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t+1)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^p}{dt^p}\left[\frac{1}{n+t+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t+1)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{p}\binom{p}{k}\frac{d^{p-k}}{dt^{p-k}}\left(\frac{1}{n+t+1}\right)\frac{d^k}{dt^k}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t+1)}\right)$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{p}\binom{p}{k}(-1)_{p-k}\frac{1}{(n+t+1)^{p-k+1}}\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(n+t+1)}.$$ q.e.d.

Theorem 3.2: *If the assumptions are the same as Theorem 1, then*

$$\left({}_0I_x^\alpha\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\sum_{k=0}^{p}\binom{p}{k}(-1)_{p-k}\frac{1}{(2n+t+1)^{p-k+1}}\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}. \tag{25}$$

Proof Since $\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]$

$$=\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha 2n}\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(2n+t+1)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}.$$

(26)

It follows from differentiation under fractional integral sign that

$$\frac{d^p}{dt^p}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\frac{d^p}{dt^p}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\frac{d^p}{dt^p}\left[E_\alpha\left(tLn_\alpha(x^\alpha)\right)\right]\otimes_\alpha cos_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^\alpha\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]. \tag{27}$$

Therefore,

$$\left({}_0I_x^\alpha\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]$$

$$=\frac{d^p}{dt^p}\left({}_0I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha cos_\alpha(x^\alpha)\right]$$

$$=\frac{d^p}{dt^p}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!(2n+t+1)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\frac{d^p}{dt^p}\left[\frac{1}{2n+t+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\sum_{k=0}^{p}\binom{p}{k}\frac{d^{p-k}}{dt^{p-k}}\left(\frac{1}{2n+t+1}\right)\frac{d^k}{dt^k}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\sum_{k=0}^{p}\binom{p}{k}(-1)_{p-k}\frac{1}{(2n+t+1)^{p-k+1}}\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}.$$

q.e.d.

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Theorem 3.3: *If the assumptions are the same as Theorem 1, then*

$$\left({}_0I_x^{\alpha}\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\sum_{k=0}^{p}\binom{p}{k}(-1)_{p-k}\frac{1}{(2n+t+2)^{p-k+1}}\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+2)}. \quad (28)$$

Proof : Since $\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]$

$$=\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+1)}\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+1)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+t+2)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+2)}. \quad (29)$$

It follows from differentiation under fractional integral sign that

$$\frac{d^p}{dt^p}\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\frac{d^p}{dt^p}\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\frac{d^p}{dt^p}\left[E_\alpha\left(tLn_\alpha(x^\alpha)\right)\right]\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$=\left({}_0I_x^{\alpha}\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]. \quad (30)$$

Thus,

$$\left({}_0I_x^{\alpha}\right)\left[\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha p}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$=\frac{d^p}{dt^p}\left({}_0I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha t}\otimes_\alpha sin_\alpha(x^\alpha)\right]$$

$$=\frac{d^p}{dt^p}\left[\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!(2n+t+2)}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+2)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\frac{d^p}{dt^p}\left[\frac{1}{2n+t+2}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+2)}\right]$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\sum_{k=0}^{p}\binom{p}{k}\frac{d^{p-k}}{dt^{p-k}}\left(\frac{1}{2n+t+2}\right)\frac{d^k}{dt^k}\left(\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+2)}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\sum_{k=0}^{p}\binom{p}{k}(-1)_{p-k}\frac{1}{(2n+t+2)^{p-k+1}}\left(Ln_\alpha(x^\alpha)\right)^{\otimes_\alpha k}\otimes_\alpha\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha(2n+t+2)}.$$

q.e.d.

IV. CONCLUSION

In this paper, we obtain three types of fractional integrals by using differentiation under fractional integral sign based Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions. In fact, our results are generalizations of traditional calculus results. In the future, we will continue to study the problems in applied mathematics and fractional differential equations.

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